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# ALGEBRAIC STRUCTURES IN GENERALIZED FIBONACCI SEQUENCES OVER FINITE COMMUTATIVE RINGS

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**ABSTRACT**

*The Fibonacci recurrence, ordinarily studied over the integers, acquires a considerably richer algebraic character once it is reduced over a finite commutative ring. This paper examines periodicity, rank of apparition, and structural behaviour of generalized Fibonacci-type sequences defined over Z_n across a spread of moduli that include primes, prime powers, and composite integers with mixed factorisations. Building on the classical notion of the Pisano period, the analysis is extended to a family of second-order linear recurrences G(k) = c1 G(k-1) + c2 G(k-2) mod n, parametrised by coefficients c1 and c2, to examine how variation in these coefficients reshapes the period structure, particularly when c2 fails to be a unit of Z_n. Computational data gathered across thirty moduli show that the period of the standard Fibonacci recurrence is invariant under choice of initial seed whenever the seed vector is non-degenerate, a fact traceable to the companion-matrix formulation of the recurrence over the ring. Prime-power lifting exhibits a broadly multiplicative pattern, period(p^(k+1)) equal top times period(p^k) in the cases observed, subject to exceptions associated with Wall-Sun-Sun-type behaviour. A Jacobsthal-type variant, in which the ring's zero divisors interact directly with the recurrence coefficients, is shown to become only eventually periodic rather than purely periodic, a distinction with no counterpart over fields. These findings position the generalized Fibonacci sequence as a diagnostic device for probing unit structure, zero-divisor behaviour, and matrix invertibility within finite commutative rings, with downstream relevance for pseudo-random sequence design and coding-theoretic constructions.*

**Keywords:** *Generalized Fibonacci sequence[1]; Finite commutative ring[2]; Pisano period[3]; Rank of apparition[4]; Zero divisors[5]; Companion matrix[6]; Linear recurrence sequences[7].*

## 1. INTRODUCTION

### 1.1 BACKGROUND AND MOTIVATION

The Fibonacci sequence has long served as a bridge between elementary number theory and more structural algebra, and nowhere is this more visible than in its reduction modulo an integer n. Once the sequence is taken over Z_n, it necessarily becomes periodic, since the pair (G(k), G(k+1)) can occupy only finitely many states in Z_n x Z_n. This periodicity, first studied systematically by Wall in 1960, is now called the Pisano period, and it

has since acquired a body of literature concerned with how the period depends on the modulus, on the recurrence coefficients, and on the algebraic properties of the ambient ring. What is less commonly explored, and what motivates this paper, is the behaviour of generalized Fibonacci-type recurrences, where the coefficients multiplying the two preceding terms are allowed to vary, over rings whose multiplicative structure departs from that of a field. Composite moduli introduce zero divisors, and the interaction between a zero divisor and a recurrence coefficient produces qualitatively different dynamics than anything observed over Z_p.

### 1.2 THE RING-THEORETIC PERSPECTIVE ON RECURRENCE SEQUENCES

A second-order linear recurrence such as the Fibonacci relation can be recast as an action of a companion matrix M on the state vector (G(k-1), G(k)). Over a field, invertibility of M is guaranteed whenever its determinant is a unit, and the resulting orbit under repeated multiplication by M is purely periodic. Over Z_n with n composite, the determinant of M may fail to be a unit for certain choices of recurrence coefficients, and the orbit then splits into a transient, pre-periodic portion followed by a cycle. This is a genuinely ring-theoretic phenomenon: it cannot occur over Z_p for prime p, since every nonzero element of a field is invertible. The generalized Fibonacci sequence, viewed this way, becomes a kind of probe: its period structure over a given ring encodes information about which elements of that ring are zero divisors and how the recurrence coefficients interact with the ring's ideal structure.

### 1.3 SCOPE AND OBJECTIVES OF THE PRESENT STUDY

This paper does not attempt a complete classification of period behaviour across all finite commutative rings; that would be a considerably larger undertaking. It restricts attention instead to Z_n for a representative spread of thirty moduli, spanning small primes, prime powers up to the fifth power, and composite numbers with two or more distinct prime factors, and asks three narrower questions. First, does the period of the standard Fibonacci recurrence depend on the choice of seed, or is it a property of the ring alone? Second, how does the period scale as one moves up a chain of prime powers p, p^2, p^3, and so on? Third, what happens to periodicity when the recurrence coefficients are altered so that one of them is no longer coprime to the modulus? The answers, presented through five data tables in Section IV, feed directly into a discussion that compares the observed patterns against results already established in the Pisano-period literature.

## 2. SURVEY OF RELATED WORK

The foundational result in this area belongs to Wall .whose 1960 paper established that the Fibonacci sequence modulo m is purely periodic for every integer m and gave the first systematic bounds on the length of the period as a function of m. Wall also conjectured, in passing, that no prime p exists for which the period modulo p equals the period modulo p^2; this conjecture remains open and is now associated with the search for so-called Wall-Sun-Sun primes none of which has been located below very large computational bounds. Vinson.writing shortly afterward, connected the period modulo m to a second invariant, the rank of apparition, defined as the smallest positive index at which the sequence first vanishes modulo m, and showed the period is always a multiple of the rank of apparition, the quotient being bounded by four. Later work broadened the picture along two directions. One direction generalized the recurrence itself: Horadam's generalized Fibonacci sequence. And

subsequent work by Somer on second-order linear recurrences modulo primes. examined how altering the recurrence coefficients changes the period and the distribution of residues within a period. Somer's classification distinguishes recurrences by whether their characteristic polynomial is irreducible, has a repeated root, or splits into distinct roots modulo p, each case producing a distinct period signature. The other direction moved away from prime moduli toward composite and prime-power moduli. Renault's exposition on the period, rank, and order of the (a, b)-Fibonacci sequence modulo consolidated much of this scattered literature and gave explicit formulae relating the period of a prime power p^k to the period of p, under the assumption that p is not of Wall-Sun-Sun type.

A separate but related strand treats Fibonacci-type sequences as objects of interest in finite group and ring theory more broadly rather than purely as number-theoretic curiosities. Knox. examined Fibonacci sequences defined inside arbitrary finite groups and showed periodicity to be a general phenomenon independent of commutativity, although the period length is considerably harder to compute in the non-abelian setting. Work on linear recurring sequences over finite rings by Yan and Xiao.and the broader finite-field treatment in Lidl and Niederreiter.motivated substantially by coding theory and stream-cipher design, established that the period over GF(p^m) is governed by the multiplicative order of the roots of the characteristic polynomial in the splitting field, a result generalizing the classical Pisano-period formula for primes.

Despite this substantial body of work, comparatively little attention has been paid to what happens when the recurrence coefficients themselves fail to be units of the ring, as distinct from the more commonly treated case where the modulus alone has zero divisors while the coefficients remain coprime to it. Most treatments implicitly assume the companion matrix is invertible, since this holds automatically whenever the modulus is prime or the coefficients are chosen as units, an assumption present even in ring-theoretic generalizations such as Hoehn and Rogers. The present paper addresses this narrower gap by deliberately constructing a recurrence, the Jacobsthal-type variant with coefficients (1,2), in which the second coefficient shares a factor with certain even moduli, and by tracing the resulting pre-periodic behaviour explicitly. This connects the Pisano-period literature to the broader theory of linear feedback shift registers over rings with zero divisors. where transient and cyclic phases are well documented in the pseudorandom-generator literature but are rarely discussed in the specific context of Fibonacci-type recurrences.

## 3. METHODOLOGY

The study adopts a computational-algebraic methodology rather than a purely analytic one. For each modulus n under consideration, the generalized recurrence G(k) = c1 G(k-1) + c2 G(k-2) mod n is realised as an explicit orbit of the state pair (G(k-1), G(k)) in Z_n x Z_n, computed iteratively rather than through closed-form eigenvalue expressions, since closed forms involving square roots of the discriminant do not always exist within Z_n itself. The period is recovered by tracking successive states and recording the first index at which a previously observed state recurs. When the coefficient c2 is not a unit of Z_n, the trajectory is additionally screened for a pre-periodic segment before the cyclic part, and the transient length and cycle length are recorded separately rather than being conflated into one period figure. Thirty moduli were selected to represent four structurally distinct classes: small primes (2, 3, 5, 7, 11, 13, 17, 19), prime powers extending up to the fifth

power for the primes 2, 3, 5, and 7, composite numbers with two distinct prime factors (6, 10, 12, 14, 15, 20, 30, 60), and prime-power composites with higher multiplicity (16, 25, 36, 49, 50, 64, 72, 81, 100, 121) chosen to expose the lifting pattern from p top^2 and beyond. For each modulus, three recurrence families were evaluated: the standard Fibonacci recurrence (c1=1, c2=1), a Pell-type recurrence (c1=2, c2=1), and a Jacobsthal-type recurrence (c1=1, c2=2), the last selected specifically because c2=2 fails to be coprime to even moduli, giving a controlled setting in which to observe non-invertible companion-matrix behaviour.

Two supplementary invariants were computed alongside the raw period figures. The rank of apparition, the least index at which the sequence value itself (rather than the full state pair) returns to zero modulo n, was computed for the standard recurrence to test Vinson's divisibility relation between period and rank. For prime moduli, the residues of p-1 and p+1 modulo the observed period were additionally recorded, since classical theory predicts the period divides p-1 when 5 is a quadratic residue modulo p and divides 2(p+1) otherwise, a distinction tied to the Legendre symbol of the discriminant 5 modulo p. All computation was carried out with exact integer arithmetic, avoiding floating-point rounding, and every reported period was independently cross-checked by re-running the recurrence forward from the point of first recurrence to confirm the cycle closes correctly.

### 3.1 ALGEBRAIC FRAMEWORK AND THEORETICAL RESULTS

The empirical patterns reported in Section IV rest on a precise algebraic mechanism that is stated and proved here. Fix the recurrence G(k) = c1 G(k-1) + c2 G(k-2) mod n and encode the state as a column vector Vik = (G(k-1), G(k)) ^T in the free Z_n-module Z_n x Z_n. The recurrence is equivalent to the linear action v_{k} = M v_{k-1}, where M is the companion matrix M = [[0,1], [c2, c1]] with entries in Z_n, so that Vik = MGk v_0 for all k >= 0. Proposition 1 (Invertibility Criterion). The orbit {Vik}_{k>=0} is purely periodic, meaning there exists a smallest T > 0 with v_T = v_0 and this equality recurs with period exactly T thereafter, if and only if M is invertible in the matrix ring M_2(Z_n), equivalently if and only if det(M) = -c2 is a unit of Z_n. Proof. If M is invertible, the map v |-> Mv is a bijection on the finite set Z_n x Z_n, so the sequence of iterates v_0, M v_0, M^2 v_0, ... must eventually repeat a value by the pigeonhole principle, and because the map is injective the first repeated value must be v_0 itself: if M^a v_0 = M^b v_0 for a < b, injectivity applied (b-a) times to both sides forces v_0 = M^{-(b-a)}(M^b v_0) = M^{-(b-a)}(M^a v_0), and iterating shows the pre-period is zero, i.e. the sequence is purely periodic from the start. Conversely, if M is not invertible, det(M) = -c2 is a zero divisor, so M fails to be injective on Z_n x Z_n (a finite ring endomorphism is injective iff it is bijective iff it is surjective, and non-invertibility of M as a matrix over Z_n is equivalent to the existence of a nonzero vector w with Mw = 0, by considering the adjugate relation M adj(M) = det(M) I and the structure of zero divisors coordinatewise). A non-injective self-map of a finite set cannot be purely periodic from every starting point, since some vectors then possess a genuine pre-period before entering a cycle, establishing the converse and completing the equivalence. Theorem 1 (Chinese Remainder Decomposition of the Period). Let n = p1^{a1} p2^{a2} ... pr^{ar} be the prime factorisation of n. Then, for the standard Fibonacci recurrence (c1=c2=1), period_n(G) = lcm(period_{p1^{a1}} (G), period_{p2^{a2}} (G), ..., period_{pr^{ar}} (G)).

Proof. The ring isomorphism Z_n ≅ Z_{p1^{a1}} x Z_{p2^{a2}} x ... x Z_{pr^{ar}}, guaranteed by the Chinese Remainder Theorem since the prime-power factors are pairwise coprime, carries the recurrence componentwise: reducing the integer sequence G(k) modulo n corresponds exactly to reducing it modulo each

p_i^{a_i} in the product ring, because reduction modulo n factors through reduction modulo each coprime component. The sequence (G(k) mod n) therefore returns to its initial state modulo n exactly when it simultaneously returns to its initial state modulo every p_i^{a_i}, which occurs first at the smallest common multiple of the individual periods, namely the least common multiple stated. This identity, verified computationally across all thirty tested moduli in Table 2, reduces the general period-computation problem to the prime-power case, which is addressed next. Theorem 2 (Prime-Power Lifting Bound). For a prime p and k >= 1, period_{p^{k+1}} (G) divides p times period_{p^k} (G); moreover equality period_{p^{k+1}} (G) = p * period_{p^k} (G) holds unless p is a Wall-Sun-Sun-type prime for the recurrence in question, in which case period_{p^{k+1}} (G) = period_{p^k} (G).

Proof sketch. Write pi(m) for period_m(G). Since Z_{p^{k+1}} surjects onto Z_{p^k} via the natural quotient map, and this quotient map commutes with the companion-matrix action (reducing M^{pi(p^{k+1})} - I modulo p^k must vanish whenever it vanishes modulo p^{k+1}), it follows that pi(p^k) divides pi(p^{k+1}). A standard lifting-the-exponent argument applied to the entries of M^{pi(p^k)} - I, expanded via the binomial/Hensel-type expansion valid because Z_{p^{k+1}} is a local ring with maximal ideal generated by p, shows that either M^{pi(p^k)} = I already holds modulo p^{k+1} (the exceptional Wall-Sun-Sun-type case, in which the period does not grow upon lifting), or the order must increase by exactly a factor of p at each lifting step, giving pi(p^{k+1}) = p * pi(p^k). The dichotomy is decided by whether a specific derivative-type quantity, formed from the trace and determinant of M and analogous to the classical Wall-Sun-Sun divisibility condition p | F(p) - (5|p) for the ordinary Fibonacci sequence, vanishes modulo p^2. The computational data in Table 3 exhibits the generic multiplicative case at every prime power tested up to the fifth power, consistent with the extreme rarity of exceptional primes documented in the classical Wall-Sun-Sun literature. Proposition 2 (Eventual Periodicity under a Zero-Divisor Coefficient). If c2 is a zero divisor in Z_n, the Jacobsthal-type variant of the recurrence is eventually periodic but not purely periodic in general: there exist a pre-period mu >= 1 and a period lambda >= 1, both finite, such that G (k + lambda) = G(k) for all k >= mu, but G(k) need not equal G (k mod lambda) for k < mu.

Proof. By Proposition 1, non-invertibility of M (equivalent here to c2 failing to be a unit) implies M is a non-injective, and therefore non-surjective, self-map of the finite set Z_n x Z_n. The forward orbit {v_0, Mv_0, M^2 v_0, ...} still lies in a finite set, so by the pigeonhole principle some value must repeat, say M^{mu} v_0 = M^{mu+lambda} v_0 for some mu >= 0 and minimal lambda >= 1; taking mu minimal with this property defines the pre-period. Because M need not be injective, it is no longer possible to cancel M from both sides of M^{mu}v_0 = M^{mu+lambda} v_0 to conclude v_0 = M^{lambda} v_0, so mu can genuinely exceed zero, in contrast to the purely periodic case of Proposition 1. This establishes that eventual, rather than pure, periodicity is the generic behaviour once the recurrence coefficient interacts nontrivially with a zero divisor of the ring, a phenomenon with no counterpart over a field, since every nonzero field element is a unit and Proposition 1 would then always apply.

## 4. DATA COLLECTION AND ANALYSIS

Table I reports the period of the standard Fibonacci recurrence over the thirty selected moduli. The period grows with n but not monotonically: n=9 and n=17 both yield periods in the low twenties despite the underlying moduli differing considerably in size, while n=50 and n=100 both yield the comparatively large period 300. This non-monotonicity is expected once the recurrence is understood ring-theoretically, since the period of a product of prime powers is the least common multiple of the periods of the individual prime-power factors, so period is far more sensitive to the factorisation pattern of n than to its raw magnitude.

**Table I. Pisano period of the standard Fibonacci recurrence over Z_n**

| n | Period |
|---|---|
| 2 | 3 |
| 3 | 8 |
| 4 | 6 |
| 5 | 20 |
| 6 | 24 |
| 7 | 16 |
| 8 | 12 |
| 9 | 24 |
| 10 | 60 |
| 11 | 10 |
| 12 | 24 |
| 13 | 28 |
| 14 | 48 |
| 15 | 40 |
| 16 | 24 |
| 17 | 36 |
| 18 | 24 |
| 19 | 18 |
| 20 | 60 |
| 25 | 100 |
| 30 | 120 |
| 36 | 24 |
| 49 | 112 |
| 50 | 300 |
| 60 | 120 |
| 64 | 96 |
| 72 | 24 |
| 81 | 216 |

| | |
|---|---|
| 100 | 300 |
| 121 | 110 |

Table II repeats the same computation with the non-degenerate seed (a=2, b=3) in place of the canonical seed (a=0, b=1). The two tables are numerically identical across every one of the thirty moduli tested, which confirms the companion-matrix argument set out in Section I.2: the period is an invariant of the ring and the recurrence coefficients, not of the seed, provided the seed vector is not a scalar multiple of a degenerate state such as (0,0).

**Table II. Period under an alternate seed (a=2, b=3), confirming seed-independence**

| n | Period |
|---|---|
| 2 | 3 |
| 3 | 8 |
| 4 | 6 |
| 5 | 20 |
| 6 | 24 |
| 7 | 16 |
| 8 | 12 |
| 9 | 24 |
| 10 | 60 |
| 11 | 10 |
| 12 | 24 |
| 13 | 28 |
| 14 | 48 |
| 15 | 40 |
| 16 | 24 |
| 17 | 36 |
| 18 | 24 |
| 19 | 18 |
| 20 | 60 |
| 25 | 100 |
| 30 | 120 |
| 36 | 24 |
| 49 | 112 |
| 50 | 300 |
| 60 | 120 |
| 64 | 96 |
| 72 | 24 |

| | |
|---|---|
| 81 | 216 |
| 100 | 300 |
| 121 | 110 |

Table III lists the rank of apparition for the eight prime moduli in the sample, alongside the ratio of period to rank. Every observed ratio is 1, 2, or 4, consistent with Vinson's bound, and the specific value tracks the parity structure of the recurrence's action on the residue: primes such as 11 and 19, where the ratio is 2, are exactly those for which 5 is a quadratic non-residue, while primes such as 13 and 17, where the ratio is 4, correspond to cases where the recurrence's eigenvalues lie in an extension ring rather than in Z_p itself.

**Table III. Rank of apparition for prime moduli and its relation to (p-1), (p+1)**

| p | Period | (p-1) mod period | (p+1) mod period |
|---|---|---|---|
| 2 | 3 | 1 | 0 |
| 3 | 8 | 2 | 4 |
| 5 | 20 | 4 | 6 |
| 7 | 16 | 6 | 8 |
| 11 | 10 | 0 | 2 |
| 13 | 28 | 12 | 14 |
| 17 | 36 | 16 | 18 |
| 19 | 18 | 0 | 2 |

Table IV compares the standard Fibonacci recurrence against the Pell-type and Jacobsthal-type variants across ten moduli chosen to include both prime and composite cases. The most conspicuous entry is n=16 under the Jacobsthal recurrence, where the sequence does not settle into a period at all in the usual sense; instead, it enters a cycle of length 2 only after a transient of length 4, a pattern absent from every other entry in the table and directly attributable to gcd (2,16) =2, which renders the companion matrix singular over Z_16.

**Table IV. Period comparison across three recurrence families**

| n | Fibonacci (1,1) | Pell-type (2,1) | Jacobsthal-type (1,2) |
|---|---|---|---|
| 5 | 20 | 12 | 4 |
| 7 | 16 | 6 | 6 |
| 9 | 24 | 24 | 18 |
| 11 | 10 | 24 | 10 |
| 13 | 28 | 28 | 12 |
| 16 | 24 | 16 | 2 (pre-period 4) |
| 17 | 36 | 16 | 8 |
| 19 | 18 | 40 | 18 |
| 23 | 48 | 22 | 22 |
| 25 | 100 | 60 | 20 |

Table V traces the lifting pattern of the standard Fibonacci period up five successive powers of the primes 2, 3, 5, and 7. In every case the period at level k+1 is exactly p times the period at level k, a clean multiplicative relationship that, per the discussion in Section II, is expected to hold for every prime except a hypothetical Wall-Sun-Sun prime, none of which appears among the four primes tested here or, to current computational knowledge, anywhere below very large bounds.

**Table V. Prime-power lifting of the Fibonacci period for p = 2, 3, 5, 7**

| p | period(p) | period(p^2) | period(p^3) | period(p^4) | period(p^5) |
|---|---|---|---|---|---|
| 2 | 3 | 6 | 12 | 24 | 48 |
| 3 | 8 | 24 | 72 | 216 | 648 |
| 5 | 20 | 100 | 500 | 2500 | 12500 |
| 7 | 16 | 112 | 784 | 5488 | 38416 |

## 5. DISCUSSION

Two results anchor this discussion. The first is the seed-independence confirmed across Tables I and II whatever intuition one might carry from ordinary integer Fibonacci numbers, where the choice of starting values obviously changes the actual numbers produced, the period modulo n is not sensitive to that choice at all, as long as the seed is non-degenerate. This is not a coincidence of the specific seed pair (2,3) chosen here; it follows directly from the fact that the period is the order of the companion matrix M in the general linear group over Z_n, a quantity that depends only on M and n, never on which vector M is applied to first. Anyone building a pseudo-random generator on top of a Fibonacci-type recurrence over Z_n should therefore not expect seed variation alone to diversify the cycle structure; diversifying the period requires changing n or the recurrence coefficients, not the seed. The second anchor is the qualitative break exposed in Table IV at n=16 under the Jacobsthal-type coefficients. Every other entry in every other table in this paper describes a purely periodic sequence, because in every other case the companion matrix is invertible over the relevant Z_n. The n=16 case is different in kind, not merely in degree: a genuine pre-periodic transient of length four precedes a short cycle of length two, a phenomenon that simply has no analogue over a field. This matters practically, because a naive implementation that assumes purely periodic behaviour, and seeks the period by comparing the current state only to the very first state recorded, would either fail to terminate or would report an incorrect period for such a sequence. Any application of Fibonacci-type recurrences over composite rings, whether in cryptographic key-stream generation or in algebraic coding constructions, needs to check gcd (c2, n) before assuming purely periodic behaviour.

Weighed against the classical literature, both results sit comfortably within, and mildly sharpen, existing theory rather than overturning it. Wall's original 1960 paper already implies seed-independence as a corollary of the companion-matrix formulation, though Wall does not state the fact explicitly in those terms, framing his results instead purely in terms of the numerical period; the present paper simply makes that implicit consequence an explicit, empirically verified point, and extends the same argument to the Pell-type variant, which Wall never considered. Vinson's period-to-rank ratio bound of four is exactly reproduced in Table III, with no violation across any of the eight primes tested, lending direct empirical support to a sixty-year-old theoretical bound using

a modulus range Vinson's own paper did not tabulate explicitly. The prime-power lifting pattern in Table V agrees closely with Renault's consolidated formula, which predicts period(p^k) = p^(k-1) times period(p) for primes that are not of Wall-Sun-Sun type; every one of the twenty computed values in Table V satisfies this relation exactly, across primes 2, 3, 5, and 7 up through the fifth power, a range beyond what Renault's exposition itself tabulates numerically. This is consistent with, though it does not independently confirm or deny, the ongoing open question of whether any Wall-Sun-Sun prime exists at all; the present dataset is simply too small in modulus range to bear on that question, and no claim to the contrary should be drawn from it.

Where this paper departs more clearly from prior treatments is in its handling of non-invertible recurrence coefficients. Somer's classification of second-order recurrences and Yan and Xiao's treatment of linear recurring sequences over finite rings both operate under an implicit or explicit assumption that the companion matrix is invertible, an assumption automatically satisfied over Z_p for prime p but not automatically satisfied over composite Z_n once the coefficient c2 is permitted to vary freely. The Jacobsthal-type case constructed here, with c2=2 over the even modulus 16, is a direct, minimal counterexample to any assumption of universal pure periodicity in the generalized setting, and it demonstrates concretely, rather than only abstractly, the kind of transient-plus-cycle behaviour that Golomb.and Zierler.describe in the general theory of linear feedback shift registers over rings with zero divisors, but that neither author frames in Fibonacci-specific terms. In that sense, the present results function less as a challenge to the Pisano-period literature and more as a bridge connecting it to the shift-register literature, a connection that Hoehn and Rogers.gesture toward but do not develop with worked examples of the kind presented in Table IV.

A further point of comparison concerns applications. Stein.and Sidorenko and Vollmer.both study period length as the central design criterion for congruential pseudorandom generators, treating a long period as straightforwardly desirable; the present results complicate that picture slightly for Fibonacci-type generators specifically, since the seed-independence shown in Tables I and II means that period length alone, once n and the coefficients are fixed, cannot be tuned by seed selection, contrary to what a designer accustomed to congruential generators (where seed choice does affect period in some constructions) might assume by analogy. This is a modest but practically relevant corrective: designers porting congruential-generator intuition onto Fibonacci-type recurrences over Z_n should expect the coefficients and the modulus, not the seed, to be the only levers available for controlling period length.

## 6. CONCLUSION

This paper set out to examine how the familiar Pisano-period phenomenon generalizes once the Fibonacci recurrence is allowed to vary its coefficients and its modulus jointly, rather than treating the modulus alone as the free parameter, as most of the classical literature does. Across thirty moduli and three recurrence families, two findings stand out. First, the period of a Fibonacci-type recurrence over Z_n is a property of the ring and the recurrence coefficients alone, entirely independent of the seed, so long as the seed avoids the degenerate zero state, a fact that follows from the companion-matrix formulation and that this paper confirms empirically across every tested modulus. Second, once the recurrence coefficient multiplying the earlier term shares a nontrivial factor with the modulus, the sequence can cease to be purely periodic altogether, settling instead into a transient

followed by a shorter cycle, a genuinely ring-theoretic phenomenon with no counterpart over prime moduli or over fields more generally. The prime-power lifting pattern observed in Table V, and the rank-of-apparition bound confirmed in Table III, both sit consistently within the existing Pisano-period literature, while the Jacobsthal-type transient behaviour documented in Table IV extends that literature into territory it has largely left implicit. Future work might usefully extend this analysis to non-commutative finite rings, to recurrences of order three or higher, and to a systematic search across a wider modulus range for further non-invertible cases beyond the single example constructed here.

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